Lecture 22

MTH 161

THE DEFINITE INTEGRAL

We saw that the limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x \right]$$

arose when we compute area.

It actually arises in various other areas, and it's worth studing and that we don't even need f to be positive, continuous and the subinterval don't necessarily have the same length.

In general, let f be any function defined on $[a,b]$ and divide $[a,b]$ into n smaller subintervals by choosing partition points $x_0, x_1, \dots, x_n$ so that

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

The resulting collection of subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \dots,\ [x_{n-1}, x_n]$$

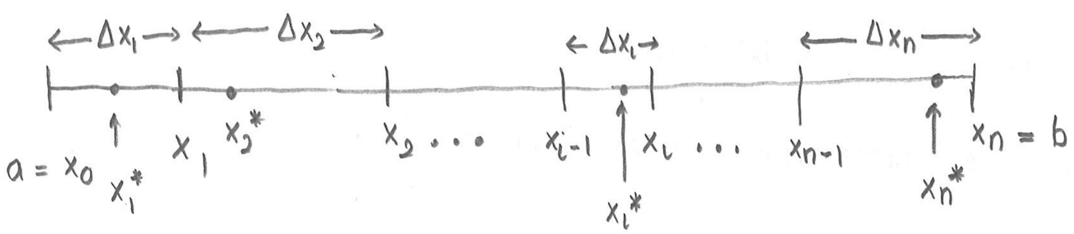
is called a Partition P of $[a,b]$

Let Δx_i for the length of the i^{th} subinterval $[x_{i-1}, x_i]$

Thus, $\Delta x_i = x_i - x_{i-1}$.

Then we choose sample points $x_1^*, x_2^*, \ldots, x_n^*$

in the subinterval with x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$.

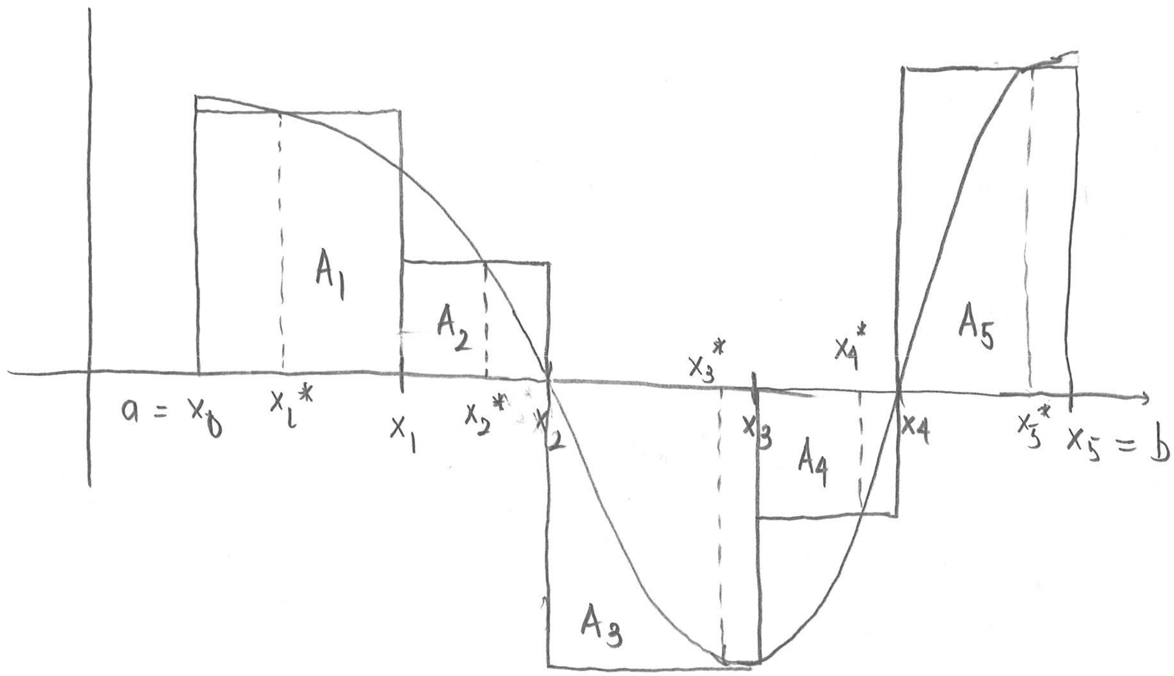


The point is that the intervals don't have the same length.

A <u>Riemann sum</u> associated with a partition P and a function f is constructed by evaluating f at the sample points, multiplying by the lengths of the corresponding subintervals, and adding

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + \cdots + f(x_n^*)\,\Delta x_n$$

The geometric interpretation of a Riemann sum is shown below.



Notice that if $f(x_i^*)$ is negative, then $f(x_i^*)\Delta x$ is negative, so we have to substract the area of the corresponding rectangle.

$$\sum_{i=1}^{5} f(x_i^*)\,\Delta x_i = A_1 + A_2 - A_3 - A_4 + A_5$$

- If we imagine all possible partitions of $[a, b]$ and all possible choices of sample points, we can think of taking the limit of all possible Riemann sums as n becomes large by analogy with the definition of area.

But because we are now allowing subintervals of different lengths, we need to ensure that all of these lengths Δx_i approach 0.

This can be done by insisting that the largest of these lengths, which we denote by $\max \Delta x_i$, approaches 0.

The result is called the definite integral of f from a to b

<u>Defn</u> Definite integral : If f is a function defined on $[a,b]$, the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided the limit exists. If it does exist, we say that f is integrable on $[a,b]$.

<u>NOTATION</u>

$$\int_{a}^{b} f(x)\,dx$$

$b \rightarrow$ upper limit.

$a \rightarrow$ lower limit

$a, b \equiv$ limits of integration

$f(x) \equiv$ integrand.

Remark

The definite integral $\int_a^b f(x)\,dx$ is a number, it does not depend on x.

In fact, we could use any letter in place of x without changing the value of the integral

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr = \int_a^b f(y)\,dy$$

Remark We defined the definite integral for an integrable function, but what is an integrable function.

Def If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$, i.e. the definite integral $\int_a^b f(x)\,dx$ exists.

Thm If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad , \text{where } \Delta x = \frac{b-a}{n}$$

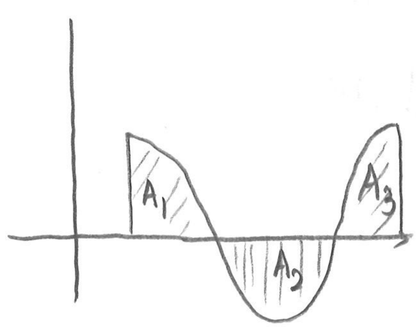
and $x_i = a + i \cdot \Delta x$

NOTE

If f is positive, then the Riemann sum can be interpreted as sum of areas of approximating rectangles.

So, $\displaystyle\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b.

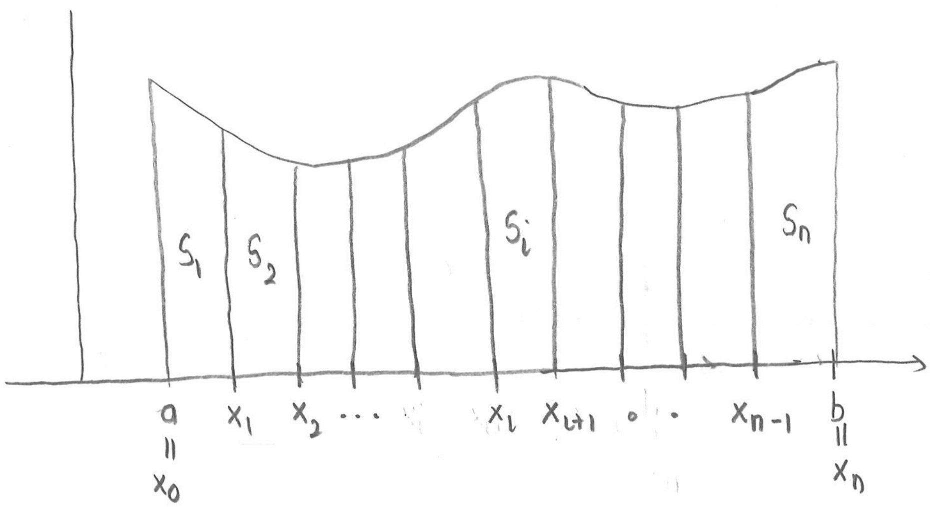
If f is both positive and negative, the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negative of the areas of the rectangles that lies below the x-axis.



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

Easier Approach

Recall We showed last class that if f is continuous, positive function on an interval $[a,b]$, divide the area under the curve into n equal strips.



We divide $[a,b]$ into n equal subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ each of interval $\Delta x = \dfrac{b-a}{n}$. This resulting collection of subintervals is called a partition P. In general for a partition, subintervals don't have to be equal length

Then the right endpoints of each interval are

$$x_1 = x_0 + \Delta x = a + \Delta x$$

$$x_2 = x_1 + \Delta x = a + \Delta x + \Delta x = a + 2\Delta x$$

$$x_3 = x_2 + \Delta x = a + 2\Delta x + \Delta x = a + 3\Delta x$$

$$\vdots$$

$$x_i = a + i \cdot \Delta x$$

Then approximating the i^{th} strip S_i by the rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint.

Then $R_n = f(x_1)\Delta x + f(x_2).\Delta x + \dots + f(x_n)\Delta x$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

We found that $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i).\Delta x$

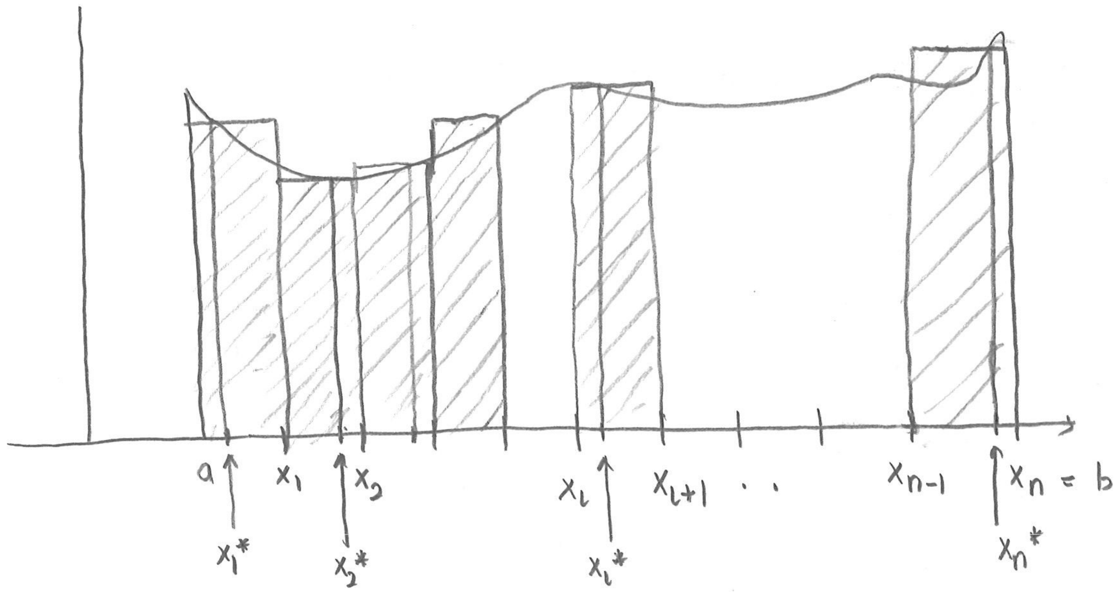
If f is continuous, we can approximating the area using left endpoints (height of the rectangle is the value of f at the left endpt of interval)

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1}).\Delta x$$

$$= \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=1}^{n} f(x_{i+1})\Delta x$$

For a continuous function.

Instead of using left or right endpoints, we can just use the height of the i^{th} rectangle to be the value of f at any number x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$.



Then $A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

All these given us the same value.

- This limit actually arises in various other areas, and it's worth studying and that we don't even need f to be positive, continuous and subintervals don't necessarily have the same length.

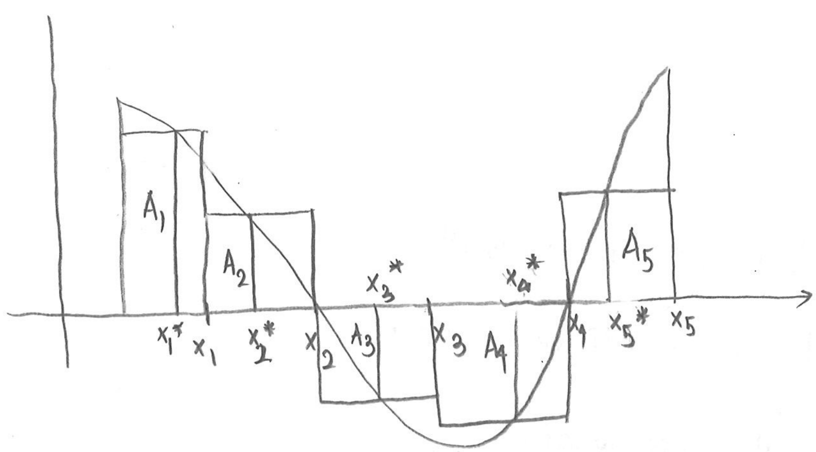
  not going to worry about it.

A Riemann sum associated with a partition P, and a function f is constructed by evaluating f at the sample points, multiplying by lengths of the corresponding subintervals:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$ ⟶ In general for Riemann sum, subintervals don't have to be the same length

The geometric interpretation of a Riemann sum

In general, f does not have to be positive.



Area is not negative.

$$\sum f(x_i^*)\,\Delta x = A_1 + A_2 - A_3 - A_4 + A_5$$

If $f(x_i^*)$ is negative, then $f(x_i^*)\,\Delta x$ is negative, so we have to substract the area of the corresponding rectangle.

Definition of Definite Integral

If f is a function defined on $[a,b]$,

the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \text{ , provided the limit exists.}$$

If the limit does exist, we say that f is integrable on $[a,b]$.

NOTATION

$$\int_{a}^{b} f(x)\,dx$$

b → upper limit

a → Lower limit

a, b = limits of integration

$f(x)$ ≡ integrand

Remark

$\int_a^b f(x)\,dx$ is a number, it does not depend on x.

We could replace x by any letter without changing the answer.

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt = \int_a^b f(\varphi)\,d\varphi$$

We defined definite integral for an integrable function and our definition of integrable function was that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x \text{ exists}.$$

We would like to say for what functions this limit exists, are integrable.

<u>Thm</u> If f is continuous on $[a,b]$ <u>or</u>

if f has only finite number of jump discontinuities,

Then f is integrable on $[a,b]$. i.e $\int_a^b f(x)\,dx$ exists.

Actually, we can define definite integral using right endpoints.

<u>Thm</u> If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) . \Delta x$$

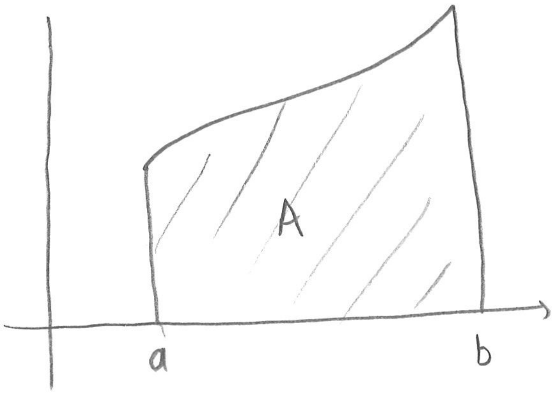
where

$$\Delta x = \frac{b-a}{n}, \text{ and } x_i = a + i\,\Delta x$$

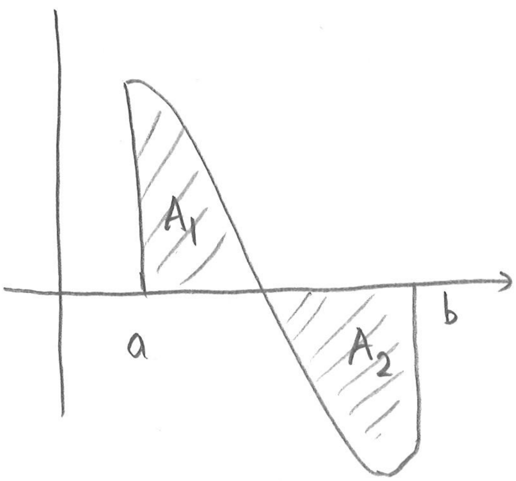
We can rewrite it as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i . \Delta x)\, \Delta x$$

Remark



$$\int_a^b f(x)\,dx = A$$



$$\int_a^b f(x)\,dx = A_1 - A_2 .$$

In order to evaluate definite integral, we will need to know some properties of the sigma notation

1) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

The remaining formula are simple rules for working with sigma notation :

$$\sum_{i=1}^{n} c = c \cdot n \longrightarrow (4)$$

5) $$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

6) $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

7) $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Ex Find $\int_0^3 (x^3 - 6x)\, dx$.

Soln $f(x) = x^3 - 6x$. So f is continuous on $[0,3]$, hence integrable.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

From, $$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x) \cdot \Delta x$$

$$f(a+i\cdot\Delta x) = f\left(0+i\cdot\frac{3}{n}\right)\cdot\frac{3}{n}$$

$$= f\left(\frac{3i}{n}\right)\cdot\frac{3}{n} = \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\cdot\frac{3}{n}$$

$$= \left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]\cdot\frac{3}{n} = \frac{81i^3}{n^4} - \frac{54i}{n^2}$$

Then, $\displaystyle\int_0^3 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{81i^3}{n^4} - \frac{54i}{n^2}\right]$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{81i^3}{n^4} - \sum_{i=1}^{n}\frac{54i}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n}i^3 - \frac{54}{n^2}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{[n(n+1)]^2}{4} - \frac{54}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left(\left[\frac{81}{4}\cdot\frac{[n(n+1)]^2}{n^4}\right] - \frac{54}{2}\cdot\frac{n(n+1)}{n^2}\right)$$

$$= \lim_{n \to \infty} \frac{81}{4} - 27$$

$$= -\frac{27}{4} = -6.75$$

Properties of the integral

Suppose all the following integrals exists

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,dx = c(b-a) \text{, where } c \text{ is a constant}$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

This property tells us how to combine integrals, or split integrals.

Ex

If $\int_0^{10} f(x)\,dx = 7$, $\int_0^{8} f(x)\,dx = 3$. Find $\int_8^{10} f(x)\,dx$

Sol By above property,

$$\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$$

$$3 + \int_8^{10} f(x)\,dx = 7 \Rightarrow \boxed{\int_8^{10} f(x)\,dx = 4}$$

Comparision property of integrals

1) If $f(x) \geqslant 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geqslant 0$

2) If $f(x) \geqslant g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$

3) If $m \leqslant f(x) \leqslant M$ for $a \leq x \leq b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

Example Use property 8 to estimate $\int_1^4 \sqrt{x}\,dx$

Soln Since $\sqrt{x}$ is an increasing function, it's abs min on $[1,4]$ is $m = f(1) = 1$ and its abs max on $[1,4]$ is $M = f(4) = 2$.

Thus,

$$1(4-1) \leqslant \int_a^b f(x)\,dx \leqslant 2(4-1)$$

$$3 \leqslant \int_a^b f(x)\,dx \leqslant 6$$